

LIFTING UNITS MODULO EXCHANGE IDEALS AND C^* -ALGEBRAS WITH REAL RANK ZERO

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ABSTRACT. Given a unital ring R and a two-sided ideal I of R , we consider the question of determining when a unit of R/I can be lifted to a unit of R . For the wide class of separative exchange ideals I , we show that the only obstruction to lifting invertibles relies on a K -theoretic condition on I . This allows to extend previously known index theories to this context. Using this we can draw consequences for von Neumann regular rings and C^* -algebras with real rank zero.

INTRODUCTION

The problem of lifting units from a quotient of a ring R modulo a two-sided ideal I has been of interest in several instances. The first extension of the classical index theory for Fredholm operators on a Hilbert space was directed to von Neumann algebras (see [9], [10] and [29]). The class of self-injective rings and that of Rickart C^* -algebras satisfying certain comparability conditions were considered by Menal and Moncasi (see [26]). The general case for Rickart C^* -algebras was studied by Ara in [2]. In all of the above cases, the extent to which a unit from a quotient R/I can be lifted to a unit of R is measured by a condition of K -theoretic nature, namely the vanishing of the connecting index map in K -Theory.

Our aim here is to consider the class of exchange ideals of unital rings, which is known to contain both (not necessarily unital) von Neumann regular rings and C^* -algebras with real rank zero. In fact, the exchange C^* -algebras are exactly those having real rank zero (see Theorem 7.2 in [6]). A second unifying principle on which we will rely is that of separative cancellation of finitely generated projective modules, which can be regarded as a weak cancellation property. Separative unital exchange rings provide a framework in which a number of outstanding open problems are known to have solutions (see [5], [6]). Moreover, this weak cancellation condition holds widely (for instance, for all the known classes of regular rings - see [6], and also for the known classes of extremally rich C^* -algebras - see [15]); it is therefore regarded as a condition that might hold for all exchange rings.

Our main objective is to prove that if I is a separative exchange ideal of a unital ring R , then the index is the only obstruction to lifting units modulo I , thus providing a common setting in which the results mentioned above can be handled. Hence we derive some consequences for both regular rings and C^* -algebras with real rank zero (and their multiplier algebras). In order to develop our index theory, we benefit from results and techniques from [7], where a detailed analysis of elementary transformations on invertible matrices over *unital* exchange rings was carried out. Our present context is, however, different in that we deal with invertible matrices

Key words and phrases. Exchange ring, C^* -algebra, real rank, refinement, separativity, K -Theory, index.

Partially supported by the TMR EU-Programme, by the DGEIC (Spain) and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

over unital rings, which are diagonal modulo an exchange ideal. It is remarkable that, in a different direction, it has been established by Brown and Pedersen ([15], Theorem 5.2) that the index is also the only obstruction to lifting invertibles modulo separative, extremally rich ideals of unital C^* -algebras.

We now fix some notations. As a general rule, R will stand for a unital ring, whereas we shall use I to denote a nonunital ring, generally sitting inside R as a two-sided ideal. An *elementary matrix* is a matrix of the form $\mathbf{1} + re_{ij}$, where $\mathbf{1}$ is an identity matrix, e_{ij} is one of the usual matrix units (with $i \neq j$), and $r \in R$. We will denote by $E_n(R)$ the subgroup of $GL_n(R)$ generated by the elementary matrices. If $x, y \in M_n(R)$, then we use $x \oplus y$ to denote the matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, and we will denote by 1_n the unit of $M_n(R)$.

1. PRELIMINARY RESULTS

Let M be a right R -module. We say that M satisfies the *finite exchange property* (see [17]) if for every right R -module A and any decompositions

$$A = M' \oplus N = A_1 \oplus \dots \oplus A_n$$

with $M' \cong M$, there exist submodules $A'_i \subseteq A_i$ (which are in fact direct summands, by the modular law) such that

$$A = M' \oplus A'_1 \oplus \dots \oplus A'_n.$$

Following [32], we say that R is an *exchange ring* provided that R_R satisfies the finite exchange property. This notion is right-left symmetric (see [32], Corollary 2). In [28], Theorem 2.1, it is proved that M has the finite exchange property if and only if $\text{End}(M)$ is an exchange ring. Also, a useful ring-theoretic characterization was provided independently by Goodearl and Nicholson:

Lemma 1.1. ([23], Theorem 2.1 in [28]) *A unital ring R is an exchange ring if and only if for every element $a \in R$ there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. \square*

This characterization motivated the notion of an exchange ring for rings without unit (see [3]). Namely, a (possibly non-unital) ring I is said to be an *exchange ring* if for each $x \in I$, there exist an idempotent $e \in I$ and elements $r, s \in I$ such that $e = xr = x + s - xs$. As proved in Lemma 1.1 in [3], the ring I is exchange if and only if, whenever $x \in I$ and R is a unital ring containing I as a two-sided ideal, then there exists an idempotent $e \in xI$ such that $1 - e \in (1 - x)R$. Of course, the notions of unital exchange and non-unital exchange agree if the ring I has a unit.

Since we will usually have a non-unital exchange ring I which is an ideal of a unital ring R , we will adopt the terminology in [3] and say that, in this context, “ I is an exchange ideal of R ”.

The class of exchange rings is pleasantly large: it includes regular rings, π -regular rings, semiperfect rings (which are exactly the semilocal exchange rings), right self-injective rings (see Remark 2.9 (b) in [7]) and C^* -algebras with real rank zero (by Theorem 7.2 in [6]). It is closed under natural ring constructions. For example, if I is an exchange ideal of R and $e \in R$ is an idempotent, then eIe is an exchange ideal of eRe ([3], Proposition 1.3 and also [28], Proposition 1.10). Also, if I is an exchange ring, then $M_n(I)$ is an exchange ring for all $n \geq 1$ (by Theorem 1.4 in [3]). The behaviour of exchange rings under extensions is characterized by

an idempotent-lifting condition ([3], Theorem 2.2): if I is an ideal of a (possibly non-unital) ring L , then L is exchange if and only if I and L/I are exchange and idempotents can be lifted modulo I .

Let I be an ideal of a unital ring R . We denote by $FP(I, R)$ the class of all finitely generated projective right R -modules P such that $P = PI$, and we define $V(I)$ to be the set of all isomorphism classes of elements from $FP(I, R)$. Note that $V(I)$ becomes an abelian monoid under the operation $[P] + [Q] = [P \oplus Q]$. Even though $V(I)$ involves the unital ring R , it can be shown that it only depends on the ring structure of I (see, for example, [26], or also [31]). It will be sometimes convenient to use an alternate description of $V(I)$ via idempotents (see [31]), so we identify $V(I)$ with the set of equivalence classes of idempotents in $M_\infty(I)$, the non-unital ring of $\omega \times \omega$ matrices with only finitely many nonzero entries from I . We will use $[e]$ to indicate the class of e in $V(I)$. Viewing I inside R , we can also identify $V(I)$ with $\{[e] \in V(R) \mid e \text{ an idempotent in } M_\infty(I)\}$.

Let M be an (abelian) monoid. We can order M by using the so-called algebraic ordering: for $x, y \in M$, write $x \leq y$ provided that there exists $z \in M$ such that $x + z = y$. If S is a submonoid of M with the property that if $x \leq y$ and $y \in S$, then $x \in S$, then S is called an *order-ideal* of M . For example, if I is a two-sided ideal of a ring R , then $V(I)$ is an order-ideal of $V(R)$. Notice that if e and f are idempotents in $M_n(R)$ for some n , then $[e] \leq [f]$ is equivalent to saying that $eM_n(R)$ is isomorphic to a direct summand of $fM_n(R)$.

Let M be a monoid, and let S be an order-ideal of M . We say that M has *refinement with respect to S* if whenever $x_1 + x_2 = y_1 + y_2$ in M with at least one of x_1, x_2, y_1, y_2 in S , then there exist elements $z_{ij} \in M$ such that $x_i = z_{i1} + z_{i2}$ and $y_i = z_{1i} + z_{2i}$, for $i = 1, 2$.

Observe that if $S = M$ then we are recovering the usual definition of a refinement monoid (see, for example [6]). Also, we see from the definition that in particular, S is a refinement monoid. If I is an exchange ring, it is known that $V(I)$ is a refinement monoid (see [3], Proposition 1.5, and also [6], Proposition 1.2). If now I is an exchange ideal of a unital ring R , still some refinement persists in $V(R)$, as the following lemma shows:

Lemma 1.2. *Let I be an exchange ideal of a unital ring R . Then $V(R)$ has refinement with respect to $V(I)$.*

Proof: Let $[A] + [B] = [C] + [D]$, with $[A], [B], [C], [D] \in V(R)$, and assume, for example, that $[A] \in V(I)$, that is, $A \in FP(I, R)$. Then $\text{End}(A)$ is an exchange ring, so that A has the finite exchange property, and hence we may use the proof of Proposition 1.2 in [6]. \square

We say that an abelian monoid M is *separative* provided that whenever $a + a = a + b = b + b$ in M , then $a = b$. Equivalently, M is separative if the following weak cancellation condition holds: if $a + c = b + c$ and $c \leq na, nb$ for some n , then $a = b$ (see Lemma 2.1 in [6]). Accordingly, we call a ring R *separative* if $V(R)$ is a separative monoid (see [6]). The following lemma was stated in [6], Lemma 4.4, for full refinement monoids, and the proof used there can be used in our present context almost entirely. Since it will be an essential result later, we just indicate the major steps that lead to the conclusion.

Lemma 1.3. *Let M be a monoid and let S be a separative order-ideal such that M has refinement with respect to S . If $a + e = b + e$ for $a, b \in M$ and $e \in S$, and $e \leq na, nb$ for some n , then $a = b$.*

Proof: Since M has refinement with respect to S , and $e \in S$ with $e \leq na$, we can decompose $e = \sum_{i=1}^n e_i$, with $e_i \leq a$ for all i . Hence we may assume that $e \leq a$, and similarly $e \leq b$.

By refinement with respect to S we get decompositions $a = a_1 + a_2$, $b = a_1 + b_2$, and $e = b_2 + c_2 = a_2 + c_2$. Note that $c_2 \in S$ and that $c_2 \leq e \leq a = a_1 + a_2$, hence we may use refinement with respect to S again. It is not difficult to see that we can arrange the resulting decompositions and change notation in such a way that $c_2 \leq a_2, b_2$. Now separativity in S entails $a_2 = b_2$, so $a = b$. \square

Recall that an element x in a ring R is called *von Neumann regular* if there exists $y \in R$ such that $x = xyx$. If y can be chosen to be a unit, then x is called *unit-regular*. In this case $x = (xy)y^{-1}$ is a product of an idempotent with a unit.

Proposition 1.4. *Let I be a separative exchange ideal of a unital ring R , and let $d \in I$ be such that $dR = (1 - p)R$ and $Rd = R(1 - q)$, for some idempotents $1 - p, 1 - q \in I$. If $RpR = RqR = R$ then $[p] = [q]$ in $V(R)$. In particular, d is unit-regular.*

Proof: Note that $V(I)$ is a separative monoid and that $V(R)$ has refinement with respect to $V(I)$ by Lemma 1.2. From the outset we evidently have that $[1 - p] = [1 - q]$ in $V(I) \subseteq V(R)$. Also, since $RpR = RqR = R$, we get that $[1 - p] \leq n[p], n[q]$ for some $n \in \mathbb{N}$. Now, in $V(R)$, we have that $[p] + [1 - p] = [q] + [1 - p]$. By Lemma 1.3, we get that $[p] = [q]$ in $V(R)$, so that $pR \cong qR$. Now, note that $R/dR = R/(1 - p)R \cong pR \cong qR = r.\text{ann}(d)$, whence d is unit-regular, by the proof of Theorem 4.1 in [21]. \square

2. INDEX THEORY FOR EXCHANGE RINGS

Lemma 2.1. (cf. [7], Lemma 2.1) *Let I be an exchange ideal of a unital ring R , and let $e_1, e_2 \in R$ be idempotents such that $e_1 \in I$. Then, there exists an idempotent $e \in e_1R + e_2R$ such that $[e_i] \leq [e]$ in $V(R)$, for all i . In particular, $ReR = Re_1R + Re_2R$.*

Proof: Considering that $\text{End}(e_1R) = e_1Re_1 = e_1Ie_1$ is a unital exchange ring, we have that e_1R has the finite exchange property, and hence we get decompositions $e_2R = A \oplus B$ and $(1 - e_2)R = A' \oplus B'$ such that $R = e_1R \oplus A \oplus A'$. Then, choose $e \in R$ such that $eR = e_1R \oplus A$, and proceed as in the proof of Lemma 2.1 in [7]. \square

In the next technical lemmas, we will be involved with performing several elementary row and column operations on an invertible 2×2 matrix over a ring R . These operations will follow the lines of 2.3-2.7 in [7]. However, our ring R won't be exchange, so we cannot apply the results in [7] directly, and hence some different procedure is needed.

Let I be a two-sided ideal in a unital ring. We shall denote by $\pi : R \rightarrow R/I$ the natural quotient map. For any $n > 1$, let $E_n(I)$ be the subgroup of $E_n(R)$ generated by the elementary matrices $1_n + re_{ij}$ for $r \in I$ and $i \neq j$. Note that $\pi(\epsilon) = 1_n$ for all $\epsilon \in E_n(I)$. Observe also that multiplying a matrix $\alpha \in M_n(R)$ on the left or right by any matrices from $E_n(I)$ does not change $\pi(\alpha)$.

Lemma 2.2. *Let I be an exchange ideal of a unital ring R . Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ such that $b, c \in I$.*

- (a) There exist $\beta \in E_2(I)$ and an idempotent $1 - h \in I$ such that $\alpha\beta = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, with $c' \in Rc$, $c'R = (1 - h)R$, $d'R = hR$ and $RhR = R$.
- (b) There exist $\gamma \in E_2(I)$ and an idempotent $1 - k \in I$ such that $\gamma\alpha = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$, with $b'' \in bR$, $Rb'' = R(1 - k)$, $Rd'' = Rk$ and $RkR = R$.

Proof: (a). First note that the row (c, d) is right unimodular, so $cR + dR = R$. Hence there exist $x, y \in R$ such that $cx + dy = 1$. Now, since $c \in I$ and I is exchange, there exists an idempotent $e \in cxR \subseteq cR$ such that $1 - e \in dR$. Write $e = cr$, with $re = r$ and $1 - e = ds$, with $s(1 - e) = s$. As in [7], Lemma 2.4, we multiply α on the right by $\alpha_1\alpha_2$, where $\alpha_i \in E_2(I)$ are defined as follows:

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ -sc & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & -rd \\ 0 & 1 \end{pmatrix}.$$

We thus obtain as last row: $(ec, (1 - e)d)$. As in the proof of Lemma 2.4 in [7] we get $ec \in cRc$, $(1 - e)d \in dRd$, and $R = ecR \oplus (1 - e)dR$. Set $w = ec + (1 - e)d$, and note that $ewr + (1 - e)ws = 1$, hence $RwR = R$. Similar to the proof of Corollary 2.5 in [7], we want to put w in the $(2, 2)$ position. By the exchange property again (applied to $ewr \in I$), we get an idempotent $f \in ewrR \subseteq ewR$ such that $1 - f \in (1 - ewr)R \subseteq (1 - e)wR$. Write $f = eww_1$, with $w_1f = w_1$, and $1 - f = (1 - e)ww_2$, with $w_2(1 - f) = w_2$, and set $f_1 = ww_1e \in wR$, and $f_2 = ww_2(1 - e) \in wR$. Note that f_i are idempotents with $f_1 \in I$, and that $f_1R \cong fR$, whereas $f_2R \cong (1 - f)R$. By Lemma 2.1, there is an idempotent $g \in f_1R + f_2R \subseteq wR$ such that $RgR = Rf_1R + Rf_2R$. Since $Rf_1R = RfR$ and $Rf_2R = R(1 - f)R$, we see that $RgR = R$.

Write $g = ww'$, for some $w' \in R$, and multiply $\alpha\alpha_1\alpha_2$ on the right by $\beta_1\beta_2$, where $\beta_i \in E_2(I)$ are defined as:

$$\beta_1 = \begin{pmatrix} 1 & rc \\ 0 & 1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 1 & 0 \\ -w'ec & 1 \end{pmatrix}.$$

This gives as last row: $((1 - g)ec, w)$. At this point we start the first part of the procedure again with the current last row, so that after right multiplication by two matrices $\gamma_i \in E_2(I)$ for $i = 1, 2$, we get as last row (c', d') , with $c' \in (1 - g)ecR(1 - g)ec \subseteq (1 - g)Rc$, and $R = c'R \oplus d'R$. According to the direct sum decomposition, we see that there exists an idempotent $1 - h \in I$ such that $c'R = (1 - h)R$ and $d'R = hR$. Since $g(1 - h)R = gc'R = 0$ and $RgR = R$, we conclude that $RhR = R$. Finally, set $\beta = \alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2$.

(b). Since $b \in I$ and I is exchange, we can perform the transpose version of the process carried out in (a) to the last column. Hence we obtain matrices α'_i , β'_i and γ'_i in $E_2(I)$, for $i = 1, 2$, such that after left multiplication by $\gamma := \gamma'_2\gamma'_1\beta'_2\beta'_1\alpha'_2\alpha'_1$ we get a matrix $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$ and an idempotent $1 - k \in I$ satisfying the desired properties. \square

Lemma 2.3. *Let I be a separative exchange ideal of a unital ring R , and let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R)$ such that $b, c \in I$ and $d - 1 \in I$. Then there exist units $a', u \in GL_1(R)$, and matrices β, γ and ϵ in $E_2(R)$ such that*

$$\gamma\alpha\beta(1 \oplus u^{-1})\epsilon = a' \oplus 1,$$

and $\pi(a') = \pi(au^{-1})$. In particular, $[\alpha] = [a'u]$ in $K_1(R)$.

Proof: We remark again that $\pi(\alpha)$ remains unchanged after right or left multiplication by matrices from $E_2(I)$. Thus we apply Lemma 2.2 (a), and without loss of generality there is an idempotent $1 - h \in I$ such that $cR = (1 - h)R$, $dR = hR$ and $RhR = R$.

Now we proceed as in the proof of Lemma 2.7 in [7], so that we move c to the $(1, 2)$ position. This is achieved after right and left multiplication by the signed permutation matrix $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which is a product of three elementary matrices. Hence, we get:

$$\alpha' := \sigma \alpha \sigma = \begin{pmatrix} -d & c \\ b & -a \end{pmatrix}.$$

By Lemma 2.2 (b), there exist $\gamma' \in E_2(I)$ and an idempotent $1 - q \in I$ such that $\gamma' \alpha' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, with $b' \in cR = (1 - h)R$, $Rb' = R(1 - q)$ and $RqR = R$. Note that $\pi(a') = -1$. Since b' is regular, there exists an idempotent $1 - p \in I$ such that $(1 - p)R = b'R$. Using that $h(1 - p)R = hb'R \subseteq hcR = 0$ and $RhR = R$, we conclude that $RpR = R$. Hence, by Proposition 1.4 (and since I is separative, by hypothesis), b' is unit-regular. Write $b' = fu$, where $f^2 = f \in I$ and $u \in GL_1(R)$.

Now move the element b' to the $(2, 2)$ position, multiplying $\gamma' \alpha'$ on the left by σ^{-1} . Then we multiply $\sigma^{-1} \gamma' \alpha'$ on the right by $\lambda = \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}$. The matrix we obtain is $\alpha'' := \begin{pmatrix} r & s \\ t & f \end{pmatrix}$, where $r \in I$, and $\pi(t) = -1$.

We want to use now Lemma 2.3 in [7], in order to get 1 in the $(2, 2)$ position and zeros elsewhere in the last row and column. First, multiply α'' on the right by $\epsilon_1 = \begin{pmatrix} 1 & 0 \\ -ft & 1 \end{pmatrix} \in E_2(I)$, so we get as last row $((1 - f)t, f)$. Since this row is right unimodular, we have that $(1 - f)tR + fR = R$, and hence $(1 - f)tR = (1 - f)R$. Choose $v \in R(1 - f)$ such that $(1 - f)tv = (1 - f)$. Since $\pi(t) = -1$, we have that $\pi(v) = -1$. After right multiplication by $\epsilon_2 = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$, our second row becomes $((1 - f)t, 1)$. Denote by z' the element in the $(1, 2)$ position of the resulting matrix.

Finally, set $\mu_1 = \begin{pmatrix} 1 & -z' \\ 0 & 1 \end{pmatrix}$ and $\mu_2 = \begin{pmatrix} 1 & 0 \\ -(1 - f)t & 1 \end{pmatrix}$, the last routine matrices. Observe that

$$\pi(\mu_1) = \begin{pmatrix} 1 & \pi(-z') \\ 0 & 1 \end{pmatrix} \text{ and that } \pi(\mu_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

After multiplying $\alpha'' \epsilon_1 \epsilon_2$ on the left by μ_1 and on the right by μ_2 , we get a matrix of the form $a' \oplus 1$, where $a' \in R$. We need to compute $\pi(a')$. By all the calculations performed so far, we have that:

$$\begin{aligned} \pi(a') \oplus 1 &= \pi(\mu_1 \sigma^{-1} \gamma' \sigma \alpha \sigma \lambda \epsilon_1 \epsilon_2 \mu_2) = \\ &= \begin{pmatrix} 1 & \pi(-z') \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi(a) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi(u^{-1}) \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

After computing the right-hand side of the above equality, we obtain:

$$\pi(a') \oplus 1 = \begin{pmatrix} \pi(au^{-1}) & \pi(au^{-1}) - \pi(z') \\ 0 & 1 \end{pmatrix},$$

whence we get that $\pi(a') = \pi(au^{-1})$, as desired. \square

Let R be a ring, and let I be a two-sided ideal. We define the *Fredholm elements relative to I* as the set $F(I, R) = \pi^{-1}(GL_1(R/I))$.

Note that $F(I, R)$ is a multiplicative subsemigroup of R , such that $GL_1(R) + I \subseteq F(I, R)$. Observe that if R has stable rank one, then $GL_1(R) + I = F(I, R)$. Denote by $\delta : K_1(R/I) \rightarrow K_0(I)$ the connecting map in algebraic K -Theory, and recall that there is an exact sequence

$$K_1(R) \xrightarrow{\pi_1} K_1(R/I) \xrightarrow{\delta} K_0(I) \rightarrow K_0(R) \rightarrow K_0(R/I)$$

([31], Theorem 2.5.4). As usual, cf. [14], 6.1, we define the *index map* as the semigroup homomorphism

$$\text{index} : F(I, R) \rightarrow K_0(I),$$

given by the rule $\text{index}(x) = \delta([\pi(x)])$.

We are now in position to prove our main result:

Theorem 2.4. *Let I be a separative exchange ideal of a unital ring R . Let $x \in R$ be a Fredholm element relative to I . Then there exists $y \in GL_1(R)$ such that $x - y \in I$ if and only if $\text{index}(x) = 0$. In this case, for any $\alpha \in K_1(R)$ that is mapped to $[\pi(x)]$, we may find $y \in GL_1(R)$ such that $[y] = \alpha$ and $x - y \in I$.*

Proof: Assume that $\pi(x)$ can be lifted to a unit of R . Then there exists $y \in GL_1(R)$ such that $x = y + b$, with $b \in I$. Hence

$$\text{index}(x) = \delta([\pi(x)]) = \delta([\pi(y)]) = \delta\pi_1([y]) = 0,$$

by exactness.

Conversely, suppose that $\text{index}(x) = 0$. Again by exactness this means that there exists $k \in \mathbb{N}$ and $y_1 \in GL_k(R)$ such that $[\pi(y_1)] = [\pi(x)]$. Hence, if $m = 2^n$ is large enough, there is $\pi(z) \in E_m(R/I)$ such that $\pi(x) \oplus 1_{m-1} = \pi(z)(\pi(y_1) \oplus 1_{m-k})$. We may clearly assume that in fact $z \in E_m(R)$. Set $w_1 := z(y_1 \oplus 1_{m-k})$, and note that $[w_1] = [y_1]$ in $K_1(R)$. Denote by (a_{ij}) the entries of w_1 , and observe that $a_{ij} \in I$ whenever $i \neq j$, that $a_{ii} - 1 \in I$ for all $i \neq 1$, and that $a_{11} - x \in I$.

We apply Lemma 2.3, replacing I and R by $M_{m/2}(I)$ and $M_{m/2}(R)$. (Notice that $M_{m/2}(I)$ is a separative exchange ideal of $M_{m/2}(R)$.) Thus we obtain matrices $u, w'_2 \in GL_{m/2}(R)$ such that $\pi(w'_2 u) = (\pi(x) \oplus 1_{m/2-1})$, and $[w_1] = [w'_2 u]$ in $K_1(R)$. Let $w_2 := w'_2 u$. A recursive procedure shows that we get $w_n \in GL_1(R)$ such that $\pi(w_n) = \pi(x)$, and $[w_n] = [y_1]$ in $K_1(R)$. \square

Remark 2.5. Notice that, according to Theorem 3.5 in [4], every ring has a largest exchange ideal with respect to the inclusion (which might be the zero ideal).

Corollary 2.6. *Let R be a (unital) separative exchange ring such that $K_0(R) = 0$, and let I be an ideal of R . Then the units of R/I lift to those of R if and only if $K_0(I) = 0$.*

Proof: Clearly, if $K_0(I) = 0$, then the connecting map δ vanishes and hence Theorem 2.4 applies. Conversely, suppose that the units of R/I lift to units of R . By [7], Theorem 2.8, the natural map $GL_1(R/I) \rightarrow K_1(R/I)$ is surjective. It follows then that the map $\pi_1 : K_1(R) \rightarrow K_1(R/I)$ in K -Theory is surjective. Therefore we get again that $\delta = 0$, and since $K_0(R) = 0$, we conclude by exactness that $K_0(I) = 0$. \square

The previous corollary applies to the case where R is a purely infinite, right self-injective ring and recovers some results of Menal and Moncasi ([26]). If R is a right self-injective ring, then by Theorem 1.22 in [21], the quotient $R/J(R)$ is regular and right self-injective, where $J(R)$ is the Jacobson radical of R . We say that a right self-injective ring is *purely infinite* if $R/J(R)$ is a purely infinite regular ring. (According to [21], a regular ring R is *purely infinite* if $R \cong R \oplus R$.) These rings are known to be exchange and separative. Further, $K_0(R) = 0$, by the proof of Corollary 3.6 in [3].

3. C^* -ALGEBRAS WITH REAL RANK ZERO

Let A be a C^* -algebra. In order to distinguish between the algebraic and the topological K_1 -groups of A , and according to more common usage, we will denote by $K_1^{\text{alg}}(A)$ the algebraic K_1 -group of A , and we shall use $K_1(A)$ to denote the topological K_1 -group (see [8], Definition 8.1.1, [33], Definition 7.1.1). It is known that there is a natural surjective homomorphism $\gamma : K_1^{\text{alg}}(A) \rightarrow K_1(A)$ (see, for example, [7]). Since idempotents in $M_n(A)$ are equivalent to projections (e.g., [8]), we may identify $V(A)$ with the abelian monoid of Murray-von Neumann equivalence classes of projections arising from $M_\infty(A)$.

Recall that a (unital) C^* -algebra A has *real rank zero* provided that every self-adjoint element can be approximated arbitrarily well by self-adjoint, invertible elements. Other characterizations, including the original definition, may be found in [12]. If A is non-unital, then A has real rank zero if and only if the minimal unitization \tilde{A} of A (see [33]) has real rank zero. Since the C^* -algebras that are exchange rings are precisely those having real rank zero, we see that if A is a C^* -algebra with real rank zero, then $V(A)$ is a refinement monoid (see also [34], Theorem 5.3).

Brown and Pedersen have introduced in [15] the concept of *weak cancellation* for C^* -algebras, meaning that if p and q are projections in A that generate the same closed ideal I of A , and $[p] = [q]$ in $K_0(I)$, then they are (Murray-von Neumann) equivalent in A . If this property holds for $M_n(A)$, for all n , then A has *stable weak cancellation*. Notice that A has stable weak cancellation if and only if A is separative. In fact, if A has real rank zero, then A has weak cancellation if and only if A is separative (that is, the property of weak cancellation is stable). This follows using Proposition 2.8 in [6] and the fact that $V(A)$ is a refinement monoid.

The property of (stable) weak cancellation is shown to hold widely within the class of extremally rich C^* -algebras (see [13]), including those that have real rank zero (see [15], Theorem 2.11). As for the case of exchange rings, there are no examples known of extremally rich C^* -algebras without weak cancellation ([15], Remark 2.12).

Let A be a C^* -algebra, and let I be a closed, two-sided ideal of A . Denote by $\partial : K_1(A/I) \rightarrow K_0(I)$ the connecting map in topological K -Theory (see, e.g., [8], Definition 8.3.1, [33], Definition 8.1.1). We then define the index of a Fredholm element x (relative to I) as $\text{index}(x) = \partial([\pi(x)])$, where $\pi : A \rightarrow A/I$ is the natural projection map. Now, since

we have that $\partial\gamma = \delta$, where $\delta : K_1^{\text{alg}}(A/I) \rightarrow K_0(I)$ is the algebraic connecting map, we see that the two possible definitions of algebraic and topological indices for Fredholm elements coincide. From the observations made, it is clear that we can apply the result in the previous section to get the following theorem (which has been independently obtained by L.G. Brown [unpublished]). We remark that for extremally rich ideals with weak cancellation the same conclusion holds, as shown in [15], Theorem 5.2.

Theorem 3.1. *Let A be a C^* -algebra and let I be a closed ideal of A with real rank zero and weak cancellation. Let $x \in A$ be a Fredholm element relative to I . Then there exists $y \in GL_1(A)$ such that $x - y \in I$ if and only if $\text{index}(x) = 0$. In this case, for any $\alpha \in K_1(A)$ that is mapped to $[\pi(x)]$, we may find $y \in GL_1(A)$ such that $[y] = \alpha$ and $x - y \in I$.*

Proof: As in the proof of Theorem 2.4, if there exists $y \in GL_1(A)$ such that $x - y \in I$, then $\text{index}(x) = 0$.

Conversely, if $\text{index}(x) = \partial([\pi(x)]) = 0$, then since $\gamma([\pi(x)]) = [\pi(x)]$ we have in fact that $\delta([\pi(x)]) = 0$, and so Theorem 2.4 applies. \square

Remark 3.2. As for the case of exchange rings (see Remark 2.5), any C^* -algebra has a largest closed ideal of real rank zero, a description of which is given in [16], Theorem 2.3.

We now give some applications to the multiplier algebras $M(A)$ of C^* -algebras A with real rank zero. Multiplier algebras of C^* -algebras are relevant objects (since they can be used, for instance, to parametrize extensions) that have been intensively studied in the last years (to cite a few examples, among many others, see [1], [20], [11], [24], [35], [22], [30]). For the basic facts concerning multipliers see, for example, Chapter 2 in [33].

The proof of the following corollary is derived entirely as the proof of Corollary 5.8 in [15], using Theorem 3.1 instead of [15], Theorem 5.2. For any unital C^* -algebra A , we use $U(A)$ to denote the unitary group of A , whereas $U_0(A)$ stands for the connected component of the identity in $U(A)$.

Corollary 3.3. *Let A be a σ -unital and stable C^* -algebra. Suppose that A has real rank zero and weak cancellation. Then there is a short exact sequence of groups*

$$0 \rightarrow U_0(M(A)/A) \rightarrow U(M(A)/A) \rightarrow K_0(A) \rightarrow 0. \quad \square$$

Corollary 3.4. *Let A be a σ -unital and stable C^* -algebra, with real rank zero and weak cancellation. Then the following are equivalent:*

- (a) *The units of $M(A)/A$ lift to units of $M(A)$;*
- (b) *$K_0(A) = 0$;*
- (c) *$U(M(A)/A)$ is connected.*

Proof: Since A is σ -unital and stable, we have that $U(M(A))$ is connected (see [27], [19] or Theorem 16.8 in [33]). Granted this, and using also Corollary 4.3.3 in [33], it is obvious that (a) \Leftrightarrow (c). Now, (b) \Leftrightarrow (c) according to Corollary 3.3. \square

Corollary 3.5. *Let A be a σ -unital (non-unital) purely infinite simple C^* -algebra. Then the units of $M(A)/A$ lift to units of $M(A)$ if and only if $K_0(A) = 0$.*

Proof: By [35], Theorem 1.2, A is stable and has real rank zero, and by Theorem 1.4 and Proposition 1.5 in [18], A has weak cancellation. Thus the result follows from Corollary 3.4. \square

We close by remarking the fact that if A is simple, σ -unital (non-unital), with real rank zero and weak cancellation, then $U(M(A))$ is connected (and hence $K_1(M(A)) = 0$). Indeed, if all projections in A are infinite, then A is purely infinite simple, hence stable ([35], Theorem 1.2 (i)), and thus $U(M(A))$ is connected. On the other hand, if there is a nonzero finite projection $p \in A$, then pAp has stable rank one, by Theorem 7.6 in [6], whence also A has stable rank one, and then $U(M(A))$ is connected, by Lemma 3.3 in [25]. Hence, for these algebras, the units of $M(A)/A$ can be lifted to units of $M(A)$ if and only if $U(M(A)/A)$ is connected.

ACKNOWLEDGEMENTS

The author is grateful to Gert Pedersen for inviting him to the Mathematics Institute of the University of Copenhagen, where this work was carried out. It is also a pleasure to thank Ken Goodearl for many helpful comments on an earlier draft of this paper.

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